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Schrödinger equation solutions for the central field power potential energy I. $V(r) = V_0(r/a_0)^{2\nu-2}$, $\nu \geqslant 1$

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The solution of a generalized non-relativistic Schrödinger equation with radial potential energy $V(r) = V_0(r/a_0)^{2\nu-2}$ is presented. After reviewing the general properties of the radial ordinary differential equation, power series solutions are developed. The Green's function is constructed, its trace and the trace of its first iteration are calculated, and the ability of the traces to provide upper and lower bounds for the ground eigenvalue is examined. In addition, WKB-like solutions for the eigenvalues and eigenfunctions are derived. The approximation method yields valid eigenvalues for large quantum numbers (Rydberg states).

KEY WORDS: Schrödinger equation, eigenfunctions, eigenvalues, Green's function, WKB approximation, central field power potential energy

1. Introduction

The central field problem which has the potential energy $V(r) = V_0(r/a_0)^{2\nu-2}$, $\nu \geqslant 1$ and $r \in [0, \infty)$, gives the generalized Schrödinger equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} S_\sigma^{(\nu)}(\lambda; y) + \left[\lambda^2 - y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right] S_\sigma^{(\nu)}(\lambda; y) = 0, \quad y \in [0, \infty), \tag{1.1}$$

where $S(r) = rR(r)$, $y = \alpha r$, $(\alpha a_0)^{2\nu} = (2\mu a_0^2/\hbar^2)V_0$, and $\lambda^2 = (\varepsilon/V_0)(\alpha a_0)^{2\nu-2}$. $R(r)$ is the central field radial function where r is the distance separating the two particles, μ is the reduced mass of the two-particle system, ε is the energy and it is greater than zero, V_0 is an arbitrary constant used to set the potential energy scale, and a_0 may be taken as the Bohr radius or other appropriate reference distance for the problem at hand. For purposes of the discussion, $\sigma^2 \geqslant 1/4$ only so that the limit of the pseudo-potential energy in (1.1) goes to 0 or ∞ as y approaches zero guaranteeing quantized stationary states. The boundary conditions, which complete the quantum mechanical differential system, are $S_\sigma^{(\nu)}(\lambda; 0) = 0$ and $S_\sigma^{\nu}(\lambda; \infty) = 0$. $S_\sigma^{(\nu)}(\lambda; y)$ is compatible with the boundary conditions for particular values of λ only. In this paper, solutions of equation (1.1), compatible with the boundary conditions, will be designated as eigenfunctions and their values of λ will be called eigenvalues to avoid confusing them with general solutions

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which have the form of eigenfunction problems. λ will be designated as λ_n, for simplicity, or $\lambda(\nu, \sigma, n)$, $n \in \{0\} \cup \mathbb{N}$. Equation (1.1) has not only numerical solutions, but also exact analytical eigenfunctions and eigenvalues although standard lists of such solutions, such as Polyanin and Zaitsev [1], do not include it.

For three-dimensional quantum problems, $\sigma = l^* + 1/2$, where l^* is the azimuthal or orbital angular momentum quantum number; for two-dimensional quantum problems, $\sigma = m$, where m is the magnetic quantum number ($S(y) = y^{1/2}R(y)$, $R(y)$ is the radial function); and for one-dimensional quantum problems, $\sigma^2 = 1/4$ (the potential energy contains the absolute value of y, if the problem calls for y to vary over the interval $(-\infty, \infty)$).

Equation (1.1) has several solutions for particular values of ν, which help in the analysis of the general problem, and they are expressed in terms of well-known functions:

(a) When $\nu = 1$, the solutions of equation (1.1) are

$$\begin{aligned} \lambda^2 > 1: &\quad S^{(1)}_{\pm\sigma}(\lambda; y) = y^{1/2} J_{\pm\sigma}\left(\sqrt{\lambda^2 - 1}\, y\right), \\ \lambda^2 = 1: &\quad S^{(1)}_{\sigma}(1; y) = y^{\sigma+1/2}, \quad S^{(1)}_{-\sigma}(1; y) = y^{-\sigma+1/2}, \\ \lambda^2 < 1: &\quad S^{(1)}_{\pm\sigma}(\lambda; y) = y^{1/2} I_{\pm\sigma}\left(\sqrt{1 - \lambda^2}\, y\right) \quad \text{or} \\ &\quad S^{(1)}_{\sigma}(\lambda; y) = y^{1/2} K_{\sigma}\left(\sqrt{1 - \lambda^2}\, y\right), \end{aligned} \tag{1.2}$$

where the I's, J's, and K's are various Bessel functions [2,3] (the definitions of Bessel functions are taken from [2]). The solutions are not quantized. When $\lambda^2 > 1$, the solutions have a countable but infinite number of zeroes in $y \in [0, \infty)$; when $\lambda^2 \leqslant 1$, the solutions have no more than one zero on $[0, \infty)$. When $\sigma = l^* + 1/2$ and $\lambda^2 > 1$, the problem is equivalent to an unbound rotating two-particle system.

(b) When $\nu = 2$, the eigenfunctions are $S^{(2)}_{\sigma}(\lambda_n; y) = y^{\sigma+1/2} \mathrm{e}^{-(1/2)y^2} L^{(\sigma)}_n(y^2)$, where $L^{(\sigma)}_n(y^2)$ is a Laguerre polynomial [3], and the eigenvalues are

$$\lambda^2(2, \sigma; n) = 4n + 2\sigma + 2, \quad n \in \{0\} \cup \mathbb{N}. \tag{1.3}$$

(The eigenfunctions are not normalized.) For $\sigma = l^* + 1/2$, we obtain the spherical harmonic oscillator problem.

(c) When $\nu = \infty$, the eigenfunctions are

$$\begin{aligned} 0 \leqslant y < 1: &\quad S^{(\infty)}_{\sigma}(\lambda_n; y) = y^{1/2} J_{\sigma}(\lambda_n y), \\ 1 \leqslant y < \infty: &\quad S^{(\infty)}_{\sigma}(\lambda_n; y) = 0. \end{aligned} \tag{1.4}$$

(The eigenfunctions are not normalized.) Since $y^{2\nu-2}$ is discontinuous at $y = 1$, continuity of the eigenfunctions's derivative is lost; but the eigenvalues are determined by the remaining continuity of the eigenfunctions at $y = 1$, i.e., $J_\sigma(\lambda_n) = 0$, therefore, the eigenvalues, $\lambda(\infty, \sigma, n)$, $n \in \{0\} \cup \mathbb{N}$, are the zeroes of the σth Bessel function. To good approximation for large n,

$$\lambda(\infty, \sigma; n) \approx \left(n + \frac{\sigma}{2} + \frac{3}{4}\right)\pi. \tag{1.5}$$

For $\sigma = l^* + 1/2$, the particle-in-a-spherical-box problem is generated.

(d) When $\sigma^2 = 1/4$ and $\nu = 3/2$, the eigenfunctions are the Airy functions [2,3], which are linear combinations of Bessel functions. For $\sigma = -(1/2)$, the even eigenvalues are obtained from the equation

$$J_{2/3}\left(\frac{2}{3}\lambda_n^{3/2}\right) = J_{-2/3}\left(\frac{2}{3}\lambda_n^{3/2}\right);$$

and, for $\sigma = 1/2$, the odd eigenvalues are given by

$$J_{1/3}\left(\frac{2}{3}\lambda_n^{3/2}\right) = -J_{-1/3}\left(\frac{2}{3}\lambda_n^{3/2}\right).$$

In both cases, $n \in \{0\} \cup \mathbb{N}$. The quantum mechanical problem describes a particle moving in a one-dimensional constant force field which has a discontinuous upward step at $y = 0$.

Salter and others (see [4, and references therein]) have discussed equation (1.1) when $\sigma^2 = 1/4$ because it allows a study of the changing energy levels as ν varies continuously from 2 (the one-dimensional harmonic oscillator problem) to infinity (the particle-on-a-line problem), both classic problems studied in all introductory quantum mechanics courses. Other authors have been interested in the one-dimensional problem as a model for potential energies in important problems of physics; for example, Yukalova and Yukalov [5]. Titchmarsh [6] has carefully discussed the approximate eigenvalues of equation (1.1) when $\sigma^2 = 1/4$. By looking at the three-dimensional central field problem with generalized σ, the same variation in ν may be studied, but a better understanding of the eigenfunctions is obtained because of their intimate connection with the well-known Bessel functions.

Further inspection of equation (1.1) shows that when y is close to zero, $y^{2\nu-2} \ll \lambda^2$,

$$S_\sigma^{(\nu)}(\lambda; y) \approx y^{1/2} J_\sigma(\lambda y),$$

$J_\sigma(\lambda y)$ are ordinary Bessel functions, and when y is much greater than zero for eigenfunctions, i.e., $y^{2\nu-2} \gg \lambda^2$,

$$S_\sigma^{(\nu)}(\lambda; y) \approx y^{1/2} K_{\sigma/\nu}\left(\frac{y^\nu}{\nu}\right),$$

$K_{\sigma/\nu}(y^\nu/\nu)$ are the Bessel functions of purely imaginary argument of the second kind known as Macdonald functions, which approach zero as y becomes infinite.

It was Salter's goal to use the problem as a fertile field for beginning theorists to investigate the mathematical subtleties of the Schrödinger equation and to experience the fascinating numerical intricacies of working with ordinary differential equations. Salter's work is extended and generalized in this paper. Rather than investigate numerical results only, it is better to give students an opportunity to learn more about the properties of higher transcendental functions; therefore, an investigation of the radial equation's mathematical properties is pursued. Thus, topics discussed in this paper are: the general properties of the differential equation; the use of the Frobenius method to generate the double infinite series expansion of the differential equation's general solution (these solutions are absolutely and uniformly convergent); the construction of the Green's function which gives an integral equation solution for the eigenfunctions of equation (1.1); the use of the Green's function and its first iteration to obtain upper and lower bounds for the lowest eigenvalue; and the application of a modified WKB-method to obtain approximate solutions for the eigenvalues and eigenfunctions of the differential equation.

2. General properties of the differential equation [7]

2.1. If we let $f(y) = -\lambda^2 + y^{2\nu-2} + (\sigma^2 - 1/4)/y^2$, where $\sigma^2 > 1/4$, it has a minimum at

$$y_{\min} = \left[\frac{\sigma^2 - 1/4}{\nu - 1}\right]^{1/2\nu} \tag{2.1.1}$$

and its value at $y = y_{\min}$ is $f(y_{\min}) = -\lambda^2 + \nu y_{\min}^{2\nu-2}$. If $\lambda^2 > \nu y_{\min}^{2\nu-2}$, $f(y)$ will have zeroes at $y = a$ and $y = b$ such that $a < y_{\min} < b$. For given λ, ν, and σ, b can be calculated from the continued fraction generated by the equation

$$b^{2\nu} = \left[\frac{\lambda^2}{1 + (\sigma^2 - 1/4)/b^{2\nu}}\right]^{\nu/(\nu-1)} \tag{2.1.2}$$

and, once b is known, then a/b can be obtained from the equation

$$\left(\frac{a}{b}\right)^2 \left[\frac{1 - (a/b)^{2\nu-2}}{1 - (a/b)}\right] = \frac{\sigma^2 - 1/4}{b^{2\nu}}, \tag{2.1.3}$$

which arises from the fact that $f(a)$ and $f(b)$ are both equal to zero. The continued fraction converges rapidly. Now a can be calculated directly from the ratio a/b. In general,

$$\frac{\sqrt{\sigma^2 - 1/4}}{\lambda} < a < y_{\min} < b < \lambda^{1/(\nu-1)}.$$

For $0 \leqslant y < a$ and $b < y < \infty$, $f(y) > 0$ and the solutions of equation (1.1) are non-oscillatory and grow exponentially; for $a < y < b$, $f(y) < 0$ and the solutions of equation (1.1) are oscillatory (see [7, p. 237]).

2.2. Equation (1.1) gives

$$\lim_{y\to\infty}\frac{S''(y)}{S(y)}=\infty$$

directly. Two further forms of equation (1.1) are helpful in describing the properties of $S(y)$:

$$\frac{\mathrm{d}}{\mathrm{d}y}\big[S(y)S'(y)\big]=\big[S'(y)\big]^2+f(y)S^2(y), \tag{2.2.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{S'(y)}{S(y)}\right]=f(y)-\left[\frac{S'(y)}{S(y)}\right]^2. \tag{2.2.2}$$

When $0\leqslant y<a$ and $b<y<\infty$, $S(y)S'(y)$ is an increasing function of y (see equation (2.2.1)) and, therefore, $S^2(y)$ is concave upward. As a result, $S(y)S'(y)$ has at most one zero in these intervals. In the interval $0\leqslant y<a$, the zero will be set at $S(0)=0$ so that the boundary condition of the quantum problem is satisfied; $S(y)S'(y)$ will have no other zeroes in the interval. If $b<\beta<\infty$ and $S(\beta)=0$ (β is the greatest root of $S(y)$), $S(y)S'(y)$ is less than zero to the left of β and greater than zero to the right of β.

When $a<y<b$, $S'(y)/S(y)$ has discontinuities at the roots of $S(y)$; it is a decreasing function of y and $\ln|S(y)|$ is concave downward. If $b<\beta<\infty$ and $S(\beta)=0$, then

$$\lim_{y\to\beta^-}\frac{S'(y)}{S(y)}=-\infty\quad\text{and}\quad\lim_{y\to\beta^+}\frac{S'(y)}{S(y)}=\infty;$$

furthermore,

$$\lim_{y\to\beta}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{S'(y)}{S(y)}\right]=-\infty$$

from both sides of the discontinuity. As β becomes larger and larger without limit, we obtain

$$\lim_{y\to\infty}\frac{S'(y)}{S(y)}=-\infty.$$

When β is the largest root of $S(y)$, regardless if it is greater or smaller than b but less than infinity, $S'(y)/S(y)$ must enter the region where

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{S'(y)}{S(y)}\right]>0\quad\text{for } y\in(y_0,\infty),\ y_0>\beta,$$

and, as a result, it can be shown [8] that

$$\lim_{y\to\infty}\frac{S'(y)}{S(y)}=\infty.$$

The argument proceeds as follows:

(1) $f(y)$, $S'(y)/S(y)$ and its derivative are all continuous in the interval $y \in (\beta, \infty)$.

(2) Let N be any positive number, as large as we please, and choose y so large that $f(y) > 2N^2$.

(3) If $S'(y)/S(y)$ ever comes into the interval $-N < S'(y)/S(y) < N$, then its derivative will be greater than N^2.

(4) As a result, $S'(y)/S(y)$ will increase beyond N and will not return because it is continuous and its derivative is greater than zero when it equals N.

(5) Therefore, a number y'' exists such that $S'(y)/S(y) > N$ for every $y > y''$, which means that

$$\lim_{y\to\infty} \frac{S'(y)}{S(y)} = \infty.$$

From these results, it follows that

$$\lim_{y\to\infty} S(y)S'(y) = \lim_{y\to\infty} S^2(y) \lim_{y\to\infty} \frac{S'(y)}{S(y)} = \infty.$$

(The proof only requires $S^2(y) > 0$, which is true when $S(y)$ is a function of real variables, the important case for the quantum mechanical problem.)

Also it follows that

$$\lim_{y\to\infty} \left[S'(y)\right]^2 = \lim_{y\to\infty} \frac{S'(y)}{S(y)} \lim_{y\to\infty} S(y)S'(y) = \infty$$

when the largest zero of $S(y)$ occurs at $y = \beta$, $\beta < \infty$. We may also infer from these conditions that $\lim_{y\to\infty} S^2(y) = \infty$: in this region $S^2(y) > [S'(y)]^2/f(y)$ and, using L'Hospital's rule, it may be shown that $\lim_{y\to\infty}[S'(y)]^2/f(y) = \infty$.

If the largest zero of $S(y)$ occurs as y approaches infinity, then we may conclude, using the mean value theorem and the fact that $S'(y) \neq 0$ in $y \in (b, \infty)$, that, if $\lim_{y\to\infty} S(y) = 0$, then $\lim_{y\to\infty} S'(y) = 0$.

2.3. All oscillating solutions of equation (1.1) must satisfy the inequality

$$\lambda^2 > \nu y_{\min}^{2\nu-2}. \tag{2.3.1}$$

For given λ, ν, and σ, $S(y)$ has a finite number of zeroes because there can be no more than one zero in the intervals $0 < y < a$ and $b < y < \infty$, and the distance between two consecutive zeroes in the interval $a < y < b$ is not less than $\pi/\sqrt{-f(y_{\min})}$. Continuing to use Sturm's comparison theories, the solutions will have no more than one zero in $a < y < b$ provided

$$\lambda^2 < \frac{\pi^2}{(b-a)^2} + \nu y_{\min}^{2\nu-2}, \tag{2.3.2}$$

and they will have at least m zeroes in $a < y < b$ provided that

$$\lambda^2 > \frac{m^2\pi^2}{(b-a)^2} + \nu y_{\min}^{2\nu-2}, \quad m \in \mathbb{N}. \tag{2.3.3}$$

The second result follows from a consideration of the solutions in the interval $a < a' < y < b' < b$ and then taking advantage of appropriate inequalities.

2.4. From the Sonine–Polya theorem [9], it is clear that the sequence of maxima of $S^2(y)$ decreases in magnitude as y increases when $y < y_{\min}$ and increases for increasing y if $y_{\min} < y$.

2.5. Two independent solutions of equation (1.1) are $S_\sigma^{(\nu)}(\lambda; y)$ and $S_{-\sigma}^{(\nu)}(\lambda; y)$. (See their power series definitions in section 3, equation (3.5).) Their Wronskian is

$$W\big[S_\sigma(y), S_{-\sigma}(y)\big] = -\sin(\sigma\pi), \tag{2.5.1}$$

so that when σ is an integer, the two solutions are no longer independent. A more general second solution may be defined as follows:

$$T_\sigma^{(\nu)}(\lambda; y) = \frac{S_{-\sigma}^{(\nu)}(\lambda; y) - S_\sigma^{(\nu)}(\lambda; y)\cos(\sigma\pi)}{\sin(\sigma\pi)}, \tag{2.5.2}$$

where the Wronskian is

$$W\big[S_\sigma(y), T_\sigma(y)\big] = -1 \tag{2.5.3}$$

making these two solutions independent for all values of σ. The zeroes of the two independent solutions separate one another. Furthermore, $\lim_{y\to\infty} T(y) = \pm\infty$ and $\lim_{y\to\infty} T'(y) = \pm\infty$ for all values of λ (see section 2.2).

In general, as y approaches infinity, all the solutions of the differential equation become larger or smaller without limit, but, for particular values of λ (when λ becomes an eigenvalue), the solutions approach zero as y approaches infinity. These are the solutions, which satisfy both boundary conditions required by the quantum problem, $S(0) = 0$ and $S(\infty) = 0$. They may be made square-integrable on $[0, \infty)$ and are eigenfunctions. As λ increases in magnitude starting from an eigenvalue, the zero moves toward the left and eventually into the interval $a < y < b$. Subsequently, with ever increasing λ, a new zero will appear at $y = \infty$, giving a new eigenfunction, and the process will be repeated. $S_{-\sigma}(y)$ and $T_\sigma(y)$ display the same property of zeroes but for different sets of lambdas, of course. However, neither function generates a set of quantum mechanical eigenvalues because both functions increase without limit as y approaches zero.

A more insightful way of looking at the problem [8] is to define the following particular solution of equation (1.1):

$$U_\sigma(y) = \big[S_\sigma^2(y) + T_\sigma^2(y)\big]^{1/2} \sin\big(\phi(\infty) - \phi(y)\big), \tag{2.5.4}$$

where

$$\tan\phi(y) = \frac{S_\sigma(y)}{T_\sigma(y)} \quad \text{and} \quad \lim_{y\to 0}\phi(y) = 0,$$

without loss of generality; $\phi(\infty)$ is constant with respect to y. Alternate expressions for $U_\sigma(y)$ taking advantage of the properties of the sine function are

$$U_\sigma(y) = \sin\phi(\infty)T_\sigma(y) - \cos\phi(\infty)S_\sigma(y)$$

or

$$U_\sigma(y) = \cos\phi(\infty)T_\sigma(y)\bigl[\tan\phi(\infty) - \tan\phi(y)\bigr].$$

The second particular solution is

$$V_\sigma(y) = \bigl[S_\sigma^2(y) + T_\sigma^2(y)\bigr]^{1/2}\cos\bigl(\phi(\infty) - \phi(y)\bigr). \tag{2.5.5}$$

By taking the derivative of $\tan\phi(y)$ with respect to y and using the Wronskian (2.5.3), it is easily shown that

$$\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = \frac{1}{[S_\sigma^2(y) + T_\sigma^2(y)]} > 0, \tag{2.5.6}$$

therefore, the derivative is a continuous function because the denominator on the right-hand side of the expression above is never equal to zero. $\phi(y)$ is an increasing, continuous function of y. Also,

$$\lim_{y\to 0}\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = 0 \quad \text{and} \quad \lim_{y\to\infty}\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = 0. \tag{2.5.7}$$

Furthermore, because $U_\sigma(y)$ has a finite number of zeroes, $\phi(\infty)$ must be bounded. Now it follows straightforwardly, using the third expression for (2.5.4) and L'Hospital's rule, that

$$\lim_{y\to\infty} U_\sigma(y) = \cos\phi(\infty)\lim_{y\to\infty}\frac{1}{T'(y)} \quad \text{and} \quad \lim_{y\to\infty} U'_\sigma(y) = -\cos\phi(\infty)\lim_{y\to\infty}\frac{1}{T_\sigma(y)},$$

and therefore,

$$\lim_{y\to\infty} U_\sigma(y) = 0 \quad \text{and} \quad \lim_{y\to\infty} U'_\sigma(y) = 0. \tag{2.5.8}$$

Using a similar argument, the limits of the second particular function as y approaches infinity are

$$\lim_{y\to\infty} V_\sigma(y) = \pm\infty \quad \text{and} \quad \lim_{y\to\infty} V'_\sigma(y) = \pm\infty. \tag{2.5.9}$$

These limits are valid for all values of λ. By defining these functions, we have lost the general properties $\lim_{y\to 0} U_\sigma(y) = 0$ and $\lim_{y\to 0} U'_\sigma(y) = 0$ except for particular values of λ, the eigenvalues; however, the advantage gained is that we can study the limits as y approaches zero much more easily than the limits as y approaches infinity; and, when these limits are true, the quantum eigenvalues have been determined.

Taking the partial derivative of equation (1.1) with respect to λ^2, holding σ and ν constant, and then integrating over (y, ∞) yields

$$-U_\sigma(y)\frac{\partial U'_\sigma(y)}{\partial(\lambda^2)} + U'_\sigma(y)\frac{\partial U_\sigma(y)}{\partial(\lambda^2)} + \int_y^\infty U_\sigma^2(z)\,\mathrm{d}z = 0, \tag{2.5.10}$$

where $U'_\sigma(y) = \partial U_\sigma(y)/\partial y$. If $y = \rho$ gives a zero of $U_\sigma(y)$, i.e., $U_\sigma(\rho) = 0$ (the Greek letter ρ is used for "root"), then

$$U'_\sigma(\rho)\frac{\partial U_\sigma(\rho)}{\partial(\lambda^2)} = -\int_\rho^\infty U_\sigma^2(z)\,\mathrm{d}z \tag{2.5.11}$$

and, also, this requires $\phi(\infty) = \phi(\rho) + k\pi$, where $k \in \{0, 1, 2, \ldots, k_{\max}\}$. The following three results may be derived:

$$\begin{aligned} U'_\sigma(\rho) &= \frac{-(-1)^k}{[S_\sigma^2(\rho) + T_\sigma^2(\rho)]^{1/2}}, \\ \frac{\partial U_\sigma(\rho)}{\partial\phi(\rho)} &= -(-1)^k\big[S_\sigma^2(\rho) + T_\sigma^2(\rho)\big]^{1/2}, \\ \frac{\partial U_\sigma(\rho)}{\partial(\lambda^2)} &= (-1)^k\big[S_\sigma^2(\rho) + T_\sigma^2(\rho)\big]^{1/2}\int_\rho^\infty U_\sigma^2(z)\,\mathrm{d}z. \end{aligned} \tag{2.5.12}$$

Because

$$\frac{\partial\rho}{\partial(\lambda^2)} = -\frac{[\partial U_\sigma(\rho)/\partial(\lambda^2)]}{[\partial U_\sigma(\rho)/\partial\rho]},$$

the equations in (2.5.12) give

$$\frac{\partial\rho}{\partial(\lambda^2)} = \big[S_\sigma^2(\rho) + T_\sigma^2(\rho)\big]\int_\rho^\infty U_\sigma^2(z)\,\mathrm{d}z > 0, \tag{2.5.13}$$

which tells us ρ is an increasing function of λ^2 and the zeroes of $U_\sigma(y)$ enter the interval $[0, \infty)$ at the origin, then move to the right as λ^2 increases. Furthermore,

$$\frac{\partial\phi(\rho)}{\partial(\lambda^2)} = \frac{\partial\phi(\infty)}{\partial(\lambda^2)} = \int_\rho^\infty U_\sigma^2(z)\,\mathrm{d}z > 0, \tag{2.5.14}$$

meaning that both $\phi(\rho)$ and $\phi(\infty)$ are increasing functions of λ^2. If $\phi(\infty) < N$, $N \in \mathbb{R}$, then $U_\sigma(y)$ has a finite number of zeroes; more specifically, if

$$n\pi < \phi(\infty) \leqslant (n+1)\pi, \quad n \in \{0\} \cup \mathbb{N}, \tag{2.5.15}$$

then $U_\sigma(y)$ has n zeroes in $[0, \infty)$ as well as a zero at infinity and a possible zero at the origin for some λ. In the limit as $\phi(\infty)$ approaches $(n+1)\pi$, $U_\sigma(y)$ becomes an eigenfunction, $U_\sigma^{(\nu)}(\lambda_n; y)$, such that the limit of the function and its derivative both equal zero as y approaches zero, and the eigenfunction has n zeroes in the interval $a < y < b$ as well as two others, one at $y = 0$ and a second at $y = \infty$.

2.6. The function $U_\sigma(y)$ can be used to demonstrate the convergence of the improper integrals associated with equation (1.1), especially the integrals of the eigenfunctions. Integrating equation (2.2.1) over $[c, d]$ and then taking the limit as $d \to \infty$, gives

$$-U_\sigma(c)U'_\sigma(c) = \int_c^\infty \left\{\left[U'_\sigma(y)\right]^2 - \left[\lambda^2 - y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right]U_\sigma^2(y)\right\} \mathrm{d}y, \tag{2.6.1}$$

and because $U_\sigma(c)U'_\sigma(c)$ exists, the improper integral converges. In the limit as $\lambda \to \omega_n$, $U_\sigma(y)$ becomes an eigenfunction where $U_\sigma^{(\nu)}(\omega_n; c) = 0$. As c approaches zero, ω_n becomes λ_n and $U_\sigma^\nu(\lambda_n; y)$ becomes $S_\sigma^\nu(\lambda_n; y)$.

Once the limit has been taken, it is straightforward to observe that

$$\begin{gathered}\int_0^\infty \left[\frac{\mathrm{d}S_\sigma^{(\nu)}(\lambda_n; y)}{\mathrm{d}y}\right]^2 \mathrm{d}y,\\ \int_b^\infty \left[-\lambda_n^2 + y^{2\nu-2} + \frac{\sigma^2 - 1/4}{y^2}\right]\left[S_\sigma^{(\nu)}(\lambda_n; y)\right]^2 \mathrm{d}y,\\ \int_0^a \left[-\lambda_n^2 + y^{2\nu-2} + \frac{\sigma^2 - 1/4}{y^2}\right]\left[S_\sigma^{(\nu)}(\lambda_n; y)\right]^2 \mathrm{d}y\end{gathered} \tag{2.6.2}$$

are all positive integrals, and less than

$$\int_a^b \left[\lambda_n^2 - y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right]\left[S_\sigma^{(\nu)}(\lambda_n; y)\right]^2 \mathrm{d}y,$$

which is positive and exists (it is the sum of the first three integrals); therefore, the integrals (2.6.2) converge and exist and, as a result, the following integrals converge:

$$\int_0^\infty \left[\frac{\mathrm{d}S_\sigma^{(\nu)}(\lambda_n; y)}{\mathrm{d}y}\right]^2 \mathrm{d}y \quad \text{and} \quad \int_0^\infty \left[\lambda_n^2 - y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right]\left[S_\sigma^{(\nu)}(\lambda_n; y)\right]^2 \mathrm{d}y. \tag{2.6.3}$$

From the second integral in equations (2.6.3), we see that $\int_0^\infty [S_\sigma^{(\nu)}(\lambda_n; y)]^2\,\mathrm{d}y$ also converges.

2.7. The variation of the eigenvalues with respect to ν holding σ constant is found by taking the partial derivative with respect to ν of equation (1.1), showing that

$$\frac{\partial}{\partial y}\left[U_\sigma(y)\frac{\partial}{\partial \nu}U'_\sigma(y) - U'_\sigma(y)\frac{\partial}{\partial \nu}U_\sigma(y)\right] + \left[\frac{\partial \lambda^2}{\partial \nu} - 2y^{2\nu-2}\ln(y)\right]U_\sigma^2(y) = 0. \tag{2.7.1}$$

Integrating (2.7.1) over the interval $y \in [0, \infty)$ and taking the appropriate limits gives

$$\frac{\partial}{\partial \nu}\lambda_n^2 = 2\int_0^\infty y^{2\nu-2}\ln(y)\Big[\overline{S_\sigma^{(\nu)}(\lambda_n; y)}\Big]^2\,\mathrm{d}y, \tag{2.7.2}$$

where $\overline{S_\sigma^{(\nu)}(\lambda_n; y)}$ is the normalized eigenfunction. The quantity $y^{2\nu-2}\ln(y)$ is negative or zero on $[0, 1]$ and is positive or zero on $[1, \infty)$. Therefore, the integral needs to be divided into two parts, i.e.,

$$\frac{\partial}{\partial \nu}\lambda_n^2 = 2\int_0^1 y^{2\nu-2}\ln(y)\Big[\overline{S_\sigma^{(\nu)}(\lambda_n; y)}\Big]^2\,\mathrm{d}y + 2\int_1^\infty y^{2\nu-2}\ln(y)\Big[\overline{S_\sigma^{(\nu)}(\lambda_n; y)}\Big]^2\,\mathrm{d}y. \tag{2.7.3}$$

If the absolute value of the first integral is greater than the value of the second, then $\partial\lambda_n^2/\partial\nu < 0$ and λ_n^2 is a decreasing function of ν. If the absolute value of the first integral is less than the value of the second, then $\partial\lambda_n^2/\partial\nu > 0$ and λ_n^2 is an increasing function of ν. The absolute value of the first integral is less than $1/(2\nu-1)^2$, therefore,

$$\frac{\partial}{\partial \nu}\left[\lambda_n^2 - \frac{1}{2\nu-1}\right] > 0 \tag{2.7.4}$$

and $\lambda_n^2 - 1/(2\nu-1)$ is an increasing function of ν. Study of the limits, using equation (2.7.3), shows that

$$\lim_{\nu\to 1}\frac{\partial}{\partial \nu}\lambda_n^2 = \infty \quad \text{and} \quad \lim_{\nu\to\infty}\frac{\partial}{\partial \nu}\lambda_n^2 = 0. \tag{2.7.5}$$

As $\nu \to 1$, we obtain the first limit above because the first integral in equation (2.7.3) is bounded, the second is not. From the same equation, as $\nu \to \infty$, the second limit is obtained because both integrals become zero; the first because $y^{2\nu-2}\ln(y)$ is zero everywhere on $[0, 1]$ and the second because the eigenfunction is zero on $[1, \infty)$ (see equation (1.4)). Also, $\lim_{\nu\to 1}\lambda_n^2 = 1$, $n \in \{0\} \cup \mathbb{N}$. A limit greater than or less than 1 gives rise to contradictions with respect to the changing number of eigenfunction zeroes as $\nu \to 1$ when comparing the solutions of equation (1.1).

2.8. By comparing equation (1.1) for an eigenfunction to that of a general solution where λ has been replaced by ξ, the characteristic function at $y = c$ is

$$F(\xi; c) = \alpha' U_\sigma^{(\nu)}(\xi; c) - \alpha U_\sigma'^{(\nu)}(\xi; c), \tag{2.8.1}$$

where α' and α are constants. In the limit as ξ approaches ξ_n, $F(\xi_n; c) = 0$. It can be shown that $\partial F(\xi; c)/\partial(\xi^2)$ is not equal to zero when ξ equals ξ_n; therefore, ξ_n is a simple root of the characteristic equation. As c approaches zero, ξ_n becomes λ_n, making λ_n a simple root, also.

3. Series solution of equation (1.1)

Let $S_\sigma^{(\nu)}(\lambda; y) = (\pi\lambda y/2)^{1/2} \sum_{j=0}^{\infty} f_j(y)(y^\nu/\nu)^{2j}$, take the appropriate derivatives and substitute into equation (1.1). The following differential equation is obtained for $f_{j+1}(y)$:

$$y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} f_{j+1}(y) + \left[2\bigl(2(j+1)\nu\bigr) + 1\right] y \frac{\mathrm{d}}{\mathrm{d}y} f_{j+1}(y) + \left[\bigl(2(j+1)\nu\bigr)^2 + (\lambda y)^2 - \sigma^2\right] f_{j+1}(y) = 4\nu^2 f_j(y), \quad j \in \{0\} \cup \mathbb{N}. \tag{3.1}$$

If

$$f_j(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{c_{j,k}(\sigma,\nu)}{\Gamma(\sigma+k+1)} (\lambda y/2)^{\sigma+2k}, \quad j \in \{0\} \cup \mathbb{N},$$

such that $f_0(y) = J_\sigma(\lambda y)$ ($\Gamma(x)$ is the Gamma function [3]), then, on substitution into the preceding equation, the following recursion relation is obtained:

$$\begin{aligned}
&\left[(j+1)\nu + k + 1\right]\left[(j+1)\nu + \sigma + k + 1\right] \frac{c_{j+1,k+1}(\sigma,\nu)}{c_{j+1,0}(\sigma,\nu)} \\
&\quad = (k+1)(\sigma+k+1) \frac{c_{j+1,k}(\sigma,\nu)}{c_{j+1,0}(\sigma,\nu)} \\
&\qquad + \left[(j+1)\nu\right]\left[(j+1)\nu + \sigma\right] \frac{c_{j,k+1}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)}, \quad j,k \in \{0\} \cup \mathbb{N},
\end{aligned} \tag{3.2}$$

where

$$c_{j,0}(\sigma,\nu) = \frac{1}{j!} \frac{1}{((\sigma/\nu)+1)_j} c_{0,0}(\sigma,\nu), \quad j \in \{0\} \cup \mathbb{N}.$$

($(x)_j$ is the Pochhammer symbol [3]; a square bracket is also used for the symbol in the following equations.) It is possible to solve explicitly for the coefficients:

$$\begin{aligned}
&\frac{c_{0,k}(\sigma,\nu)}{c_{0,0}(\sigma,\nu)} = 1, \quad k \in \{0\} \cup \mathbb{N}; \quad \text{and} \\
&\frac{c_{j+1,k}(\sigma,\nu)}{c_{j+1,0}(\sigma,\nu)} = \frac{k!(\sigma+1)_k}{[(j+1)\nu+\sigma+1]_k [(j+1)\nu+1]_k} \\
&\qquad \times \sum_{n_j=0}^{k} \sum_{n_{j-1}=0}^{n_j} \cdots \sum_{n_0=0}^{n_1} \prod_{l=0}^{j} \frac{[(l+1)\nu+\sigma]_{n_l}[(l+1)\nu]_{n_l}}{(l\nu+\sigma+1)_{n_l}(l\nu+1)_{n_l}}, \quad j,k \in \{0\} \cup \mathbb{N}.
\end{aligned} \tag{3.3}$$

Dividing through the recursion relation (3.2) by ν^2 and solving for the coefficients the second time, gives the following results:

$$\frac{c_{j,0}(\sigma,\nu)}{c_{0,0}(\sigma,\nu)} = 1, \quad j \in \{0\} \cup \mathbb{N}; \quad \text{and}$$
$$\frac{c_{j,k+1}(\sigma,\nu)}{c_{0,k+1}(\sigma,\nu)} = \frac{j!\left(\frac{\sigma}{\nu}+1\right)_j}{\left[\frac{(\sigma+k+1)}{\nu}+1\right]_j\left[\frac{(k+1)}{\nu}+1\right]_j} \times \sum_{n_k=0}^{j}\sum_{n_{k-1}=0}^{n_k}\cdots\sum_{n_0=0}^{n_1}\prod_{l=0}^{k}\frac{\left(\frac{\sigma+l+1}{\nu}\right)_{n_l}\left(\frac{l+1}{\nu}\right)_{n_l}}{\left[\frac{(\sigma+l)}{\nu}+1\right]_{n_l}\left(\frac{l}{\nu}+1\right)_{n_l}}, \quad j,k \in \{0\} \cup \mathbb{N}. \tag{3.4}$$

The second form of the coefficients makes it easier to follow the change in the function as ν approaches infinity.

$S_{-\sigma}^{(\nu)}(\lambda; y)$ is also a solution of equation (1.1). The two solutions are linearly independent so long as σ is not an integer. When σ is a negative integer, the second independent solution, containing a logarithmic term, can be derived using standard methods discussed by Ince [7]. If $c_{0,0}(\sigma,\nu) = 1$, then $\lim_{y\to 0} S_{-1/2}^{(\nu)}(\lambda; y) = 1$. Finally,

$$S_{\sigma}^{(\nu)}(\lambda; y) = \left(\frac{\pi}{2}y\right)^{1/2}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{(\sigma/\nu+1)_j} \times \left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\Gamma(\sigma+k+1)}\frac{c_{j,k}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)}\left(\frac{\lambda y}{2}\right)^{\sigma+2k}\right]\left(\frac{y^\nu}{2\nu}\right)^{2j} \tag{3.5}$$

will be taken as the canonical solution of equation (1.1). Notice that λ has been omitted in the first parenthesis in comparison with the trial series at the beginning of the section in order to obtain the Wronskians (2.5.1) and (2.5.3) mentioned earlier. The series is absolutely convergent. By manipulating the recursion relation, the inequalities

$$\frac{c_{j,k}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)} - \frac{c_{j,k+1}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)} \geqslant 0, \quad j,k \in \{0\} \cup \mathbb{N},$$

and

$$\frac{c_{j,k}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)} - \frac{c_{j+1,k}(\sigma,\nu)}{c_{j+1,0}(\sigma,\nu)} \geqslant 0, \quad j,k \in \{0\} \cup \mathbb{N},$$

are established for $\sigma > -1$. It is also true that

$$1 \geqslant \frac{c_{j,k}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)} \geqslant 0, \qquad j,k \in \{0\} \cup \mathbb{N}, \quad \sigma > -1,$$

and therefore,

$$\left|S_\sigma^{(\nu)}(\lambda;\, y)\right| < \Gamma\left(\frac{\sigma}{\nu}+1\right)\left(\frac{\pi}{2}y\right)^{1/2}\left(\frac{y^\nu}{2\nu}\right)^{-\sigma/\nu} I_{\sigma/\nu}\left(\frac{y^\nu}{\nu}\right) I_\sigma(\lambda y).$$

A more straightforward method to show that the c-ratios are less than or equal to one is to use the recursion relation and argue by induction using the rows and columns of the matrix

$$\frac{c_{j,k}(\sigma,\nu)}{c_{j,0}(\sigma,\nu)}.$$

When σ is replaced by $-\sigma$, the resulting series is also absolutely convergent. In this case, if

$$m < \sigma < m+1, \quad m \in \{0\} \cup \mathbb{N},$$

and

$$p\nu < \sigma < (p+1)\nu, \quad p \in \{0\} \cup \mathbb{N},$$

then

$$\left|\frac{c_{j,k}(-\sigma,\nu)}{c_{j,0}(-\sigma,\nu)}\right| \leqslant A, \qquad j \in \{0,1,2,\ldots,p+1\}, \quad k \in \{0,1,2,\ldots,m+1\}.$$

A is the least upper bound of the c-ratio absolute values in the given rectangle of the matrix. Now the recursion relation with induction allows us to show that all the other c-ratios of the infinite matrix are less than or equal to A because the coefficients of the c-ratios are now positive. As a result, $|S^\nu_{-\sigma}(\lambda;\, y)|$ is less than a product series, which is easily shown to be absolutely convergent.

$S_\sigma^{(\nu)}(\lambda;\, y)$ and $S_{-\sigma}^{(\nu)}(\lambda;\, y)$ are uniformly convergent on all closed and bound subintervals of $(0,\infty)$ by the Weierstrass test omitting $y=0$ for the second solution.

In the limit as $\nu \to 1$, the ratio of coefficients becomes

$$\frac{c_{j,k}(\sigma,1)}{c_{j,0}(\sigma,1)} = \frac{(\sigma+1)_k}{(\sigma+j+1)_k}$$

and on substitution into $S_\sigma^{(\nu)}(\lambda;\, y)$ yields the following functions:

$$\begin{aligned}
&1.\ \lambda^2 > 1: \quad S_\sigma^{(1)}(\lambda;\, y) = \left(\frac{\pi}{2}y\right)^{1/2}\frac{\lambda^\sigma}{(\sqrt{\lambda^2-1})^\sigma} J_\sigma\left(\sqrt{\lambda^2-1}\,y\right),\\
&2.\ \lambda^2 = 1: \quad S_\sigma^{(1)}(\lambda;\, y) = \frac{\sqrt{\pi}}{\Gamma(\sigma+1)}\left(\frac{y}{2}\right)^{\sigma+1/2},\\
&3.\ \lambda^2 < 1: \quad S_\sigma^{(1)}(\lambda;\, y) = \left(\frac{\pi}{2}y\right)^{1/2}\frac{\lambda^\sigma}{(\sqrt{1-\lambda^2})^\sigma} I_\sigma\left(\sqrt{1-\lambda^2}\,y\right).
\end{aligned} \tag{3.6}$$

Except for a multiplicative constant, these functions agree with the three equations (1.2).

Table 1
Approximate eigenvalues for the ground state, $\lambda(\nu, \sigma, 0)$.

$\nu \setminus \sigma$	-0.5	0.5	1.5
2	1.0000	1.732051	2.236068
3	1.0297	1.949	2.6659
3.5	1.0499	2.022	2.801
4	1.0699	2.085	2.91
6	1.14	2.262	3.2004
8	1.193	2.2732	3.381
10	1.23238	2.459	3.5088

Table 2
Approximate eigenvalues for the first excited state, $\lambda(\nu, \sigma, 1)$.

$\nu \setminus \sigma$	-0.5	0.5	1.5
2	2.236068	2.645751	3.01
3	2.73	3.4118	4.007
4	3.012	3.863	4.607
6	3.338	4.381	5.304
8	3.537	4.676	5.701
10	3.6744	4.876	5.965

Antai Cen and Michael Schneider, summer research interns at Wabash College working with me in the summers of 1994 and 1998, respectively, have written computer programs to calculate $S_\sigma^{(\nu)}(\lambda; y)$. By searching for values of λ requiring $\lim_{y\to\infty} S_\sigma^{(\nu)}(\lambda; y) = 0$, approximate eigenvalues of equation (1.1) have been calculated for the ground state and the first excited state for some values of ν and σ (see tables 1 and 2).

Considering the one-dimensional problem where $\sigma^2 = 1/4$ and $V(y) = |y|^{2\nu-2}$ on the interval $(-\infty, \infty)$, $\lambda(\nu, -1/2, 0)$ is the ground state, $\lambda(\nu, 1/2, 0)$, $\lambda(\nu, -1/2, 1)$, and $\lambda(\nu, 1/2, 1)$ are the next three excited states. The ground states for ν equal 2, 3, 4, and 8 agree with Salter's calculations [4]. (Note that Salter's lambda is the square of lambda used in this paper and his $p = \nu - 1$.)

The solutions obtained above are analogous to the Bessel functions. It is also possible to factor out positive or negative exponential functions and then develop power series solutions. These series are analogous to confluent hypergeometric functions. When σ is a negative integer, the second solution contains a logarithmic term; this series is established by generalizing the c-ratios, i.e., let

$$f_j(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{c_{j,k}(\sigma, \nu; x)}{\Gamma(\sigma + k + 1)} \left(\frac{\lambda y}{2}\right)^{x+2k}$$

(compare with $f_j(y)$ at the beginning of the section), calculate the c-ratios as before, then the second solution is

$$\lim_{x\to -m}\left[\frac{\partial S_m^{(\nu)}(\lambda; x, y)}{\partial x}\right].$$

One must examine two cases, first, σ/ν not a negative integer and, second, σ/ν a negative integer. Finally, asymptotic series solutions can also be derived.

4. The Green's function [10]

Because equation (1.1) can be solved in terms of known functions when $\lambda = 0$, it is easy to construct the Green's function. Inspection of equation (1.1) shows that the Green's function must satisfy the partial differential equation

$$\frac{\partial^2}{\partial y^2}G(x, y) + \left[-y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right]G(x, y) = -\delta(x - y). \tag{4.1}$$

Using Bessel functions of purely imaginary argument, we obtain

$$\begin{aligned} 0 \leqslant x \leqslant y < \infty: \quad G(x, y) &= \frac{1}{\nu}x^{1/2}I_{\sigma/\nu}\left(\frac{x^\nu}{\nu}\right)y^{1/2}K_{\sigma/\nu}\left(\frac{y^\nu}{\nu}\right), \\ 0 \leqslant y \leqslant x < \infty: \quad G(x, y) &= \frac{1}{\nu}x^{1/2}K_{\sigma/\nu}\left(\frac{x^\nu}{\nu}\right)y^{1/2}I_{\sigma/\nu}\left(\frac{y^\nu}{\nu}\right), \end{aligned} \tag{4.2}$$

and the integral equation's bilinear concomitant is equal to zero at $y = 0$ and $y = \infty$; therefore, the Green's function satisfies the integral equation

$$S_\sigma^{(\nu)}(\lambda; x) = \lambda^2 \int_0^\infty G(x, y)S_\sigma^{(\nu)}(\lambda; y)\,\mathrm{d}y,$$

where the possible values for λ^2 are the eigenvalues of equation (1.1). The trace of the Green's function is

$$\begin{aligned} \int_0^\infty G(x, x)\,\mathrm{d}x &= \sum_{n=0}^\infty \frac{1}{\lambda_n^2} \\ &= \frac{(2\nu)^{2/\nu}}{4(\sigma + 1)}\frac{\Gamma\left[\frac{(\sigma+1)}{\nu} + 1\right]\Gamma\left(\frac{1}{\nu} + 1\right)\Gamma\left(-\frac{2}{\nu} + 1\right)}{\Gamma\left[\frac{(\sigma-1)}{\nu} + 1\right]\Gamma\left(-\frac{1}{\nu} + 1\right)}, \quad \nu > 2. \end{aligned} \tag{4.3}$$

The first iterated kernel

$$G_1(x, y) = \int_0^\infty G(x, z)G(z, y)\,\mathrm{d}y$$

is calculated using Meijer's G-functions [3] and the trace of the kernel is

$$\begin{aligned}\int_0^\infty & G_1(x,x)\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\frac{1}{\lambda_n^4} \\ &= \frac{\sqrt{\pi}}{16}\frac{\nu^{4/\nu}}{(\sigma+1)^2(\sigma+2)}\frac{\Gamma\left[\frac{2}{\nu}(\sigma+1)+1\right]\Gamma\left(\frac{2}{\nu}+1\right)\Gamma\left[\frac{(\sigma+2)}{\nu}+1\right]}{2^{2\sigma/\nu}\Gamma^2\left(\frac{\sigma}{\nu}+1\right)\Gamma\left[\frac{(\sigma+2)}{\nu}+\frac{1}{2}\right]}D(\sigma,\nu),\end{aligned} \tag{4.4}$$

where

$$D(\sigma,\nu) = \sum_{m=0}^{\infty}\frac{\left(\frac{\sigma}{\nu}+\frac{1}{2}\right)_m\left[\frac{(\sigma+1)}{\nu}\right]_m\left[\frac{2(\sigma+1)}{\nu}\right]_m\left(\frac{2}{\nu}\right)_m\left[\frac{(\sigma+2)}{\nu}\right]_m}{m!\left(\frac{2\sigma}{\nu}+1\right)_m\left(\frac{\sigma}{\nu}+1\right)_m\left[\frac{(\sigma+2)}{\nu}+\frac{1}{2}\right]_m\left[\frac{(\sigma+1)}{\nu}+1\right]_m}, \qquad \nu > \frac{4}{3}.$$

Notice that when $\nu = 2$, the trace of the first iterated Green's function becomes

$$\int_0^\infty G_1(x,x)\,\mathrm{d}x = \sum_{m=0}^{\infty}\frac{1}{(4m+2\sigma+2)^2},$$

where the denominators in the sum are just the squares of the eigenvalues (a happy but unexpected occurrence!).

From the traces of the Green's function, $T = \int_0^\infty G(x,x)\,\mathrm{d}x$, and the first iteration of the Green's function, $T_1 = \int_0^\infty G_1(x,x)\,\mathrm{d}x$, an upper and lower bound for the lowest eigenvalue is obtained. By inspection of the infinite series for the traces given above and expressed in terms of reciprocal powers of the eigenvalues, it can be readily shown that

$$\frac{1}{T} < \frac{1}{\sqrt{T_1}} < \cdots < \lambda_0^2 < \cdots < \frac{T}{T_1}. \tag{4.5}$$

Further iteration will improve the bounds. It will also allow bounds to be assigned to higher eigenvalues. The calculation of further iterations of the Green's function looks promising but tedious.

For the one-dimensional case, $\sigma = \pm 1/2$, representing the even and odd states respectively of the symmetric potential energy function, it is possible to calculate the bounds for the ground state and first excited state directly from the inequalities for traces T and T_1.

5. The modified WKB approximation [11]

Ordinarily the WKB method uses sines and cosines to approximate oscillating regions of differential equations and hyperbolic functions outside these regions. The turning points are taken as the points where the solution passes from an oscillating into an exponential region. At these points, the WKB solution is undefined. However, for equation (1.1) it is better to use Bessel functions for a WKB-like solution because they can be chosen so that they will become the correct solutions at y near 0 and y large. Furthermore, if $y = \lambda^{1/(\nu-1)}$ is taken as the point where the WKB-like solution is undefined, the properties of Bessel functions again offer advantages for an approximate solution.

First, let us examine an approximate solution of equation (1.1) which is bounded around the point $y = \lambda^{1/(\nu-1)}$. Expanding

$$(\lambda^2 - y^{2\nu-2}) - \frac{\sigma^2 - 1/4}{y^2}$$

in the neighborhood of $y = \lambda^{1/(\nu-1)}$ gives

$$(2\nu-2)\lambda^2\left[-\frac{\sigma^2 - 1/4}{(2\nu-2)\lambda^{2\nu/(\nu-1)}} + \left(1 - \frac{\sigma^2 - 1/4}{(\nu-1)\lambda^{2\nu/(\nu-1)}}\right)\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)\right] \quad (5.1)$$

as the first two terms of the power series expansion in $(1 - y/\lambda^{1/(\nu-1)})$. If

$$\frac{\sigma^2 - 1/4}{(\nu-1)\lambda^{2\nu/(\nu-1)}} \ll 1,$$

then to good approximation

$$\left(\lambda^2 - y^{2\nu-2}\right) - \frac{\sigma^2 - 1/4}{y^2} \approx (2\nu-2)\lambda^2\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right),$$

and equation (1.1) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}S(y) + (2\nu-2)\lambda^2\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)S(y) = 0 \quad (5.2)$$

in this neighborhood. The solutions of equation (5.2) are Airy's integrals:

$$y < \lambda^{1/(\nu-1)}: \quad S(y) = \int_0^\infty \cos\left[\frac{t^3}{3} - \frac{1}{\alpha}\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)t\right]\mathrm{d}t$$

and

$$y > \lambda^{1/(\nu-1)}: \quad S(y) = \int_0^\infty \cos\left[\frac{t^3}{3} + \frac{1}{\alpha}\left(\frac{y}{\lambda^{1/(\nu-1)}} - 1\right)t\right]\mathrm{d}t,$$

where

$$\alpha^3 = \frac{1}{(2\nu-2)\lambda^{2\nu/(\nu-1)}}.$$

Airy's integrals can be expressed in terms of Bessel functions:

$$y < \lambda^{1/(\nu-1)}:\quad S(y) = \frac{\pi}{3}\left[\frac{1}{\alpha}\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)\right]^{1/2} \times \left\{J_{-1/3}\left[\frac{2}{3\alpha^{3/2}}\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)^{3/2}\right] + J_{1/3}\left[\frac{2}{3\alpha^{3/2}}\left(1 - \frac{y}{\lambda^{1/(\nu-1)}}\right)^{3/2}\right]\right\}, \tag{5.3a}$$

$$y > \lambda^{1/(\nu-1)}:\quad S(y) = \frac{1}{3^{1/2}}\left[\frac{1}{\alpha}\left(\frac{y}{\lambda^{1/(\nu-1)}} - 1\right)\right]^{1/2} K_{1/3}\left[\frac{2}{3\alpha^{3/2}}\left(\frac{y}{\lambda^{1/(\nu-1)}} - 1\right)^{3/2}\right]. \tag{5.3b}$$

If we define $\xi_1'(y) = (\lambda^2 - y^{2\nu-2})^{1/2}$ so that $\xi_1(y) = \int_y^{\lambda^{1/(\nu-1)}} (\lambda^2 - z^{2\nu-2})^{1/2}\,\mathrm{d}z$ and $\xi_2'(y) = (y^{2\nu-2} - \lambda^2)^{1/2}$ so that $\xi_2(y) = \int_{\lambda^{1/(\nu-1)}}^{y} (z^{2\nu-2} - \lambda^2)^{1/2}\,\mathrm{d}z$, the Airy integrals can be rewritten as

$$y < \lambda^{1/(\nu-1)}:\quad S(y) = \frac{[\lambda(2\nu-2)^{1/2}\alpha^{1/2}\pi]^{1/2}}{2\cos(\pi/6)}\left(\frac{\pi}{2}\frac{\xi_1(y)}{\xi_1'(y)}\right)^{1/2} \times \left[J_{1/3}\big(\xi_1(y)\big) + J_{-1/3}\big(\xi_1(y)\big)\right], \tag{5.4}$$

$$y > \lambda^{1/(\nu-1)}:\quad S(y) = \frac{[\lambda(2\nu-2)^{1/2}\alpha^{1/2}\pi]^{1/2}}{2}\left(\frac{2}{\pi}\frac{\xi_2(y)}{\xi_2'(y)}\right)^{1/2} K_{1/3}\big(\xi_2(y)\big).$$

When y is in the neighborhood of $\lambda^{1/(\nu-1)}$, these more general forms of $S(y)$ reduce correctly to equations (5.3). In addition, they have the advantage of allowing a determination of the asymptotic values of the Airy integrals:

$$y < \lambda^{1/(\nu-1)}:\quad S(y) \to \left[\lambda(2\nu-2)^{1/2}\alpha^{1/2}\pi\right]^{1/2}\frac{1}{(\xi_1'(y))^{1/2}}\cos\left(\xi_1(y) - \frac{\pi}{4}\right),$$
$$y > \lambda^{1/(\nu-1)}:\quad S(y) \to \frac{[\lambda(2\nu-2)^{1/2}\alpha^{1/2}\pi]^{1/2}}{2}\frac{1}{(\xi_2'(y))^{1/2}}\,\mathrm{e}^{-\xi_2(y)}. \tag{5.5}$$

Second, Bessel functions can be used to construct an approximate solution to equation (1.1). The three fragments of the approximate solution listed below satisfy differential equations which reduce correctly to equation (1.1) as y approaches zero and as y approaches infinity:

$$y \to 0:\quad S(y) = A\left(\frac{\pi}{2}\frac{\xi_0(y)}{\xi_0'(y)}\right)^{1/2} J_\sigma\big(\xi_0(y)\big),$$

where $\xi_0'(y) = (\lambda^2 - y^{2\nu-2})^{1/2}$ and $\xi_0(y) = \int_0^y (\lambda^2 - z^{2\nu-2})^{1/2}\,\mathrm{d}z$;

$$y \to \lambda_{\text{Left}}^{1/(\nu-1)}: \qquad S(y) = \frac{B}{2\cos(\sigma\pi/2)}\left(\frac{\pi}{2}\frac{\xi_1(y)}{\xi_1'(y)}\right)^{1/2} \times \left[J_\sigma\left(\xi_1(y)\right) + J_{-\sigma}\left(\xi_1(y)\right)\right],$$

$$y \to \lambda_{\text{Right}}^{1/(\nu-1)} \text{ and } y \to \infty: \quad S(y) = C\left(\frac{2}{\pi}\frac{\xi_2(y)}{\xi_2'(y)}\right)^{1/2} K_{\sigma/\nu}\left(\xi_2(y)\right).$$

The fragments are connected by taking advantage of their asymptotic behavior:

$$y \to 0: \qquad S(y) \to A\frac{1}{(\xi_0'(y))^{1/2}}\cos\left(\xi_0(y) - \frac{\sigma\pi}{2} - \frac{\pi}{4}\right),$$

$$y \to \lambda_{\text{Left}}^{1/(\nu-1)}: \qquad S(y) \to B\frac{1}{(\xi_1'(y))^{1/2}}\cos\left(\xi_1(y) - \frac{\pi}{4}\right),$$

$$y \to \lambda_{\text{Right}}^{1/(\nu-1)} \text{ and } y \to \infty: \quad S(y) \to C\frac{1}{(\xi_2'(y))^{1/2}}\mathrm{e}^{-\xi_2(y)}.$$

The WKB-like solution is undefined at $y = \lambda^{1/(\nu-1)}$ but the asymptotic values for the second and third fragments are the same as those of the bounded approximate solution first studied (see equations (5.3)) except for a multiplicative constant, even though the functions now depend on the subscript σ rather than $1/3$. The agreement of the asymptotic values provides a method to connect the fragments of the WKB-like solution. They join together if

$$\int_0^{\lambda^{1/(\nu-1)}} \left(\lambda^2 - z^{2\nu-2}\right)^{1/2} \mathrm{d}z - \frac{\sigma\pi}{2} - \frac{\pi}{2} = n\pi, \quad n \in \{0\} \cup \mathbb{N}, \tag{5.6}$$

$$B = (-1)^n A \quad \text{and} \quad C = \frac{(-1)^n}{2}A.$$

Because the WKB-like solution is not square-integrable, it is not possible to normalize the eigenfunction using the standard method. Reasonable arguments for different normalization factors may be made, but a specific value for A is not crucial for many calculations. On integration of equation (5.6), an approximation for the eigenvalues is obtained:

$$\lambda(\nu,\sigma,n) = \left[\frac{\Gamma\left(\frac{1}{(2\nu-2)} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{(2\nu-2)} + 1\right)\Gamma\left(\frac{3}{2}\right)}\left(n + \frac{\sigma}{2} + \frac{1}{2}\right)\pi\right]^{1-1/\nu}, \quad n \in \{0\} \cup \mathbb{N}. \tag{5.7}$$

Notice that

$$\frac{\sigma^2 - 1/4}{(\nu-1)[\lambda(\nu,\sigma,n)]^{2\nu/(\nu-1)}} = \frac{\sigma^2 - 1/4}{(\nu-1)\left[\frac{\Gamma\left(\frac{1}{(2\nu-2)}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{(2\nu-2)}+1\right)\Gamma\left(\frac{3}{2}\right)}\left(n + \frac{\sigma}{2} + \frac{1}{2}\right)\pi\right]^2},$$

which is less than 1 even when $n = 0$ for all ν and σ. Even as ν approaches one,

$$\lim_{\nu \to 1} (\nu - 1)^{1/2} \frac{\Gamma\left(\frac{1}{2\nu-2} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2\nu-2} + 1\right)} = 2^{-1/2}.$$

The restriction for the solution in the neighborhood of $y = \lambda^{1/(\nu-1)}$ is satisfied. Notice, also, that when $\nu = 2$:

$$\lambda^2(2, \sigma, n) = 4n + 2\sigma + 2, \quad n \in \{0\} \cup \mathbb{N}, \tag{5.8}$$

which, by unexpected good fortune, is the correct value for the eigenvalues (see equation (1.3)). For $\nu = \infty$:

$$\lambda(\infty, \sigma, n) = \left(n + \frac{\sigma}{2} + \frac{1}{2}\right)\pi, \quad n \in \{0\} \cup \mathbb{N}, \tag{5.9}$$

which is a good value for the $n \gg 0$ eigenvalues of the problem (see equation (1.5)); the exact values are the zeroes of the σth Bessel function. Equation (5.9) agrees reasonably well with equation (1.5); therefore, $\lambda(\nu, \sigma, n)$ gives satisfactory results for large values of n.

6. Conclusion

Although the solution of equation (1.1) is complicated, it yields to the classical methods for the study of ordinary differential equations. The application of these methods introduces the student to their power and to their use in solving non-trivial problems. Equation (1.1) provides good practice for studying the theory of differential equations. It gives the student an opportunity to distinguish the eigenfunctions from the general solutions. The eigenfunctions become more intelligible in the context of the general solutions' properties. The Frobenius method of using infinite series often leads to divergent series. We obtain a double series, cumbersome, to be sure, but convergent. Green's functions are important parts of analysis because they provide information about the eigenvalues. Being able to calculate the first iteration is a rare occurrence. It allows us to derive upper and lower bounds for the ground state eigenvalues. The problem also introduces students to the properties of the Bessel functions and makes them an important part of the young theorist's repertoire. Most important, the student gains a comprehensive perspective for considering differential equations rather than the bits and pieces usually supplied by textbooks.

The door is open to the construction of improved approximations for $S_\sigma^{(\nu)}(\lambda; y)$, its eigenfunctions, and eigenvalues. In an ε-neighborhood of an arbitrary value of y, confluent hypergeometric functions can be used as good approximations to the solutions of equation (1.1). Numerical calculations can be guided by the analytical results obtained. Better knowledge of the series solutions of equation (1.1) makes the exploration of a host of new quantum problems more reasonable, especially power potentials in wells of finite depth become a more profitable and practical pursuit.

Now it will be possible to look much more carefully at the physics contained in the problem. For example, as the problem is set up the zero of potential energy for all values of ν is taken at the origin and we learn that $\partial(\lambda_n^2)/\partial\nu$ may be positive or negative as ν increases. If equation (1.1) is modified to read

$$\frac{\mathrm{d}^2 S_\sigma^{(\nu)}(\lambda;y)}{\mathrm{d}y^2}+\left\{\left[\lambda^2-\frac{1}{2\nu-1}\right]-\left[y^{2\nu-2}-\frac{1}{2\nu-1}\right]-\frac{\sigma^2-1/4}{y^2}\right\}S_\sigma^{(\nu)}(\lambda;y)=0, \quad (6.1)$$

we know $\lambda_n^2-1/(2\nu-1)$ is an increasing function of ν and zero of potential energy becomes different for each value of ν, taking the origin as the reference point. However, the equation tells us that the average potential energy for all values of ν on the interval [0, 1] is now zero, giving a new method of comparison for the power potential energies.

The solutions for equation (1.1) make several quantum problems more tractable:

(a) The system with a power potential in an infinite well. The potential energy is defined as

$$V(y)=y^{2\nu-2}, \quad y\in[0,y_0],$$
$$V(y)=\infty, \quad y\in(y_0,\infty), \quad S_\sigma^{(\nu)}(\lambda;y)=0.$$

The solution in the first interval is $S_\sigma^{(\nu)}(\lambda;y)$ and the boundary condition requires $S_\sigma^{(\nu)}(\lambda;y_0)=0$.

(b) The system with a power potential in a finite well, which Salter called "soft potential" problems. The potential energy is defined as

$$V(y)=y^{2\nu-2}, \quad y\in\left[0,y_0\right],$$
$$V(y)=y_0^{2\nu-2}, \quad y\in\left(y_0,\infty\right).$$

The solution in the first interval is $S_\sigma^{(\nu)}(\lambda;y)$ and in the second interval is $y^{1/2}K_\sigma(\sqrt{y_0^{2\nu-2}-\lambda^2}\,y)$, the purely imaginary Bessel function of the second kind. The boundary condition requires the two functions with multiplicative constants to be continuous at $y=y_0$ and also their derivatives using the same multiplicative constants must be continuous at $y=y_0$. The eigenvalues lie in the interval $0<\lambda^2<y_0^{2\nu-2}$.

(c) Another problem [12], where the power potential takes the form $z^{2\nu-2}$, $0\leqslant\nu\leqslant 1$, is important enough because of its relation to the hydrogen-like atom problem to merit detailed discussion on its own. By choosing the units appropriately, the radial equation becomes

$$\frac{\mathrm{d}^2T(z)}{\mathrm{d}z^2}+\left[-\kappa^2+z^{2\nu-2}-\frac{\sigma^2-1/4}{z^2}\right]T(z)=0, \quad z\in[0,\infty). \quad (6.2)$$

Letting $T(z)=z^{(1/2)(1-\nu)}U(z)$, followed by choosing $y=(\kappa z/\nu)^\nu$, transforms equation (6.2) into a special case of equation (1.1).

(d) It seems best to organize the discussion of these problems around a canonical differential equation. In order to parallel Watson's discussion of the Bessel functions, an appropriate choice for general investigation of the all of the functions is the equation analogous to Bessel's equation, i.e.,

$$\frac{\mathrm{d}^2 S_\sigma^{(\nu)}(\lambda;\, y)}{\mathrm{d}y^2} + \left[\lambda^2 + y^{2\nu-2} - \frac{\sigma^2 - 1/4}{y^2}\right] S_\sigma^{(\nu)}(\lambda;\, y) = 0. \qquad (6.3)$$

The series solutions of the equation can be derived using the same method developed for equation (1.1). The series solutions of (6.3) as well as those containing logarithms are available from the author.

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